

# Diagonals of separately absolutely continuous mappings coincide with the sums of absolute convergent series of continuous functions

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## Abstract

We prove that, for an interval  $X \subseteq \mathbb{R}$  and a normed space  $Z$  diagonals of separately absolute continuous mappings  $f : X^2 \rightarrow Z$  are exactly mappings  $g : X \rightarrow Z$  which are the sums of absolute convergent series of continuous functions.

*Key words:* absolutely continuous function, semi-continuity, Baire-one mapping

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## 1 Introduction

Let  $f : X^2 \rightarrow Z$  be a mapping. We call a mapping  $g : X \rightarrow Z$ ,  $g(x) = f(x, x)$ , *the diagonal of  $f$* .

Investigations of diagonals of separately continuous functions  $f : X^2 \rightarrow \mathbb{R}$  started in the classical work of R. Baire [1]. He showed that diagonals of separately continuous functions of two real variables are exactly Baire-one functions, i.e. pointwise limits of continuous functions. His result was generalized by A. Lebesgue and H. Hahn for real-valued functions of several real variables (see [8,9,5]).

Since the second half the XX-th century, Baire classification of separately continuous mappings and their analogues is intensively studied by many mathematicians (see [13,16,18,11,2,3]). The inverse problem on construction of separately continuous functions with a given diagonal was solved in [12]. In [15] it was shown that for any topological space  $X$  and a function  $g : X \rightarrow \mathbb{R}$  of the  $(n-1)$ -th Baire class there exist a separately continuous function  $f : X^n \rightarrow \mathbb{R}$  with the diagonal  $g$ .

In [10] the diagonal variant of the problem of Eidelheit from the famous “Scottish Book” on a composition of absolutely continuous functions was investigated. It was proved that there exists a separately absolutely continuous function  $f : [0, 1]^2 \rightarrow \mathbb{R}$  such that its partial derivatives  $f'_x$  and  $f'_y$  in the degree  $p$  are integrable on  $[0, 1]^2$  for every  $p > 1$ , and such that its diagonal  $g$  is not absolutely continuous.

The following question naturally arises.

**Question 1.1** *Find necessary and sufficient conditions on a function  $g : [0, 1] \rightarrow \mathbb{R}$  under which there is a separately absolutely continuous function  $f : [0, 1]^2 \rightarrow \mathbb{R}$  with the diagonal  $g$ .*

In the given paper we prove that for any interval  $X \subseteq \mathbb{R}$  and a normed space  $Z$  diagonals of separately absolute continuous mappings  $f : X^2 \rightarrow Z$  are exactly mappings  $g : X \rightarrow Z$  which are the sums of absolute convergent series of continuous functions.

## 2 Preliminaries

For topological spaces  $X, Y$  and  $Z$  a mapping  $f : X \times Y \rightarrow Z$  which is continuous with respect to every variable is called *separately continuous*.

Let  $X$  and  $Y$  be topological spaces. A mapping  $f : X \rightarrow Y$  is a *mapping of the first Baire class* or a *Baire-one mapping* if there exists a sequence  $(f_n)_{n=1}^\infty$  of continuous mappings  $f_n : X \rightarrow Y$  which pointwise converges to  $f$  on  $X$ .

For a metric space  $X$  by  $|\cdot - \cdot|_X$  we denote the metric on this space.

Let  $X \subseteq \mathbb{R}$  be an interval and let  $Z$  be a metric space. A mapping  $f : X \rightarrow Z$  is called *absolutely continuous* if for an arbitrary  $\varepsilon > 0$  there exists  $\delta > 0$  such that for every collection  $a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n$  of elements  $a_1, b_1, \dots, a_n, b_n \in X$  with  $\sum_{k=1}^n (b_k - a_k) < \delta$  the inequality  $\sum_{k=1}^n |f(b_k) - f(a_k)|_Z < \varepsilon$  holds. Let, moreover,  $Y$  be an interval. A mapping  $f : X \times Y \rightarrow Z$  which is absolutely continuous with respect to each variable is called *separately absolutely continuous*.

A mapping  $f : X \rightarrow Z$  has *bounded variation on an interval  $X$*  if there exists  $C > 0$  such that for any collection  $a_1 < b_1 \leq a_2 < b_2 \leq \dots \leq a_n < b_n$  of elements  $a_1, b_1, \dots, a_n, b_n \in X$  the inequality  $\sum_{k=1}^n |f(b_k) - f(a_k)|_Z \leq C$  holds. Moreover, for an interval  $X = [a, b]$  the least upper bound of all values

$\sum_{k=1}^n |f(b_k) - f(a_k)|_Z$  is called *the variation of  $f$  on  $[a, b]$* .

Let  $X \subseteq \mathbb{R}$  be an interval, let  $Z$  be a metric space, let  $f : X \rightarrow Z$  be a mapping and  $x_0 \in X$ . We say that  $f$  *has finite variation at  $x_0$*  if there exists a segment  $[a, b] \subseteq X$  such that  $[a, b]$  is a neighborhood of  $x_0$  in  $X$  and  $f$  has finite variation on  $[a, b]$ .

Let  $X$  and  $Z$  be metric spaces and  $A \subseteq X$ . A mapping  $f : X \rightarrow Z$  is *Lipschitz on a set  $A$  with a constant  $C \geq 0$*  if  $|f(x) - f(y)|_Z \leq C|x - y|_X$  for any  $x, y \in A$ . A mapping  $f : X \rightarrow Z$  is called *Lipschitz on a set  $A$*  if there exists  $C \geq 0$  such that  $f$  is Lipschitz on  $A$  with the constant  $C$ . A mapping  $f : X \rightarrow Z$  is called *Lipschitz (with a constant  $C$ )* if it is Lipschitz (with a constant  $C$ ) on  $X$ .

A function  $p : X \rightarrow \mathbb{R}$  defined on a vector space  $X$  over a field  $\mathbb{K}$  is said to be a *pseudo-norm* if for any  $x, y \in X$  and  $\lambda \in \mathbb{K}$  with  $|\lambda| \leq 1$  the following conditions hold:  $p(x) \geq 0$ , moreover,  $p(x) = 0$  iff  $x = 0$ ;  $p(\lambda x) \leq p(x)$  and  $p(x + y) \leq p(x) + p(y)$ . It is well-known that for any metrizable topological vector space  $X$  there exists a metric  $\rho$  on  $X$  which generates a topological structure on  $X$  and  $\rho(x, y) = p(x - y)$  for some pseudo-norm  $p$  on  $X$ .

Let  $X$  be a topological space and let  $Z$  be a metric space. A mapping  $f : X \rightarrow Z$  is said to be an *absolute Baire-one mapping* or a *mapping of the first absolute Baire class* if there exists a sequence  $(f_n)_{n=1}^\infty$  of continuous functions  $f_n : X \rightarrow Z$  such that  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  and  $\sum_{n=1}^\infty |f_{n+1}(x) - f_n(x)|_Z < \infty$  for every  $x \in X$ . Note that for a normed space  $Z$  a mapping  $f : X \rightarrow Z$  is an absolute Baire-one mapping if and only if  $f$  is the sum of an absolute convergent series of continuous functions. It is well-known (see [17], [6, §41 section 5, §42 section 2]) that a real valued function is the sum of an absolute convergent series of continuous functions if and only if it is the sum of a lower semicontinuous and an upper semicontinuous functions. Some properties of real-valued absolute Baire-one functions were studied by many mathematicians (see, for example, [14], [7]).

### 3 Necessary conditions on diagonals of separately absolutely continuous mappings

**Proposition 3.1** *Let  $X = \mathbb{R}$ , let  $Z$  be a metric topological vector space with the metric, generated by a pseudo-norm  $p$ , let  $f : X^2 \rightarrow Z$  be a continuous mapping with respect to the first variable and let  $\alpha : X \rightarrow (0, +\infty)$  be a continuous function. Then there exist functions  $\beta, \gamma : X \rightarrow (\frac{1}{2}\alpha(x), \alpha(x))$  and*

a continuous mapping  $g : X \rightarrow Z$  such that

$$p(g(x) - f(x, x + \beta(x))) \leq p(f(x, x + \beta(x)) - f(x, x + \gamma(x)))$$

for every  $x \in X$ .

**Proof.** For every  $x \in X$  we write  $y_x = x + \frac{3}{4}\alpha(x)$  and, taking into account that  $\alpha$  is continuous at  $x$ , we choose an open neighborhood  $U_x$  of  $x$  in  $X$  such that  $t + \frac{1}{2}\alpha(t) < y_x < t + \alpha(t)$  for every  $t \in U_x$ .

Let  $(V_i : i \in I)$  be an open locally finite refinement of  $(U_x : x \in X)$  such that for every  $x \in X$  the set  $I_x = \{i \in I : x \in V_i\}$  contains at most two elements. Let  $(\varphi_i : i \in I)$  be a partition of unity on  $X$  such that  $V_i = \varphi_i^{(-1)}((0, 1])$  for every  $i \in I$ . For every  $i \in I$  we choose such  $x_i \in X$  that  $V_i \subseteq U_{x_i}$  and let  $y_i = y_{x_i}$ . For every  $x \in X$  let  $g(x) = \sum_{i \in I} \varphi_i(x) f(x, y_i)$ . Clearly,  $g$  is continuous. Moreover, according to the choice of  $U_x$  the following condition holds:

(a)  $t + \frac{1}{2}\alpha(t) < y_i < t + \alpha(t)$  for every  $t \in V_i$ .

For every  $x \in X$  we pick  $i_x \in I_x$ . Moreover, we take  $j_x \in I_x \setminus \{i_x\}$  if  $|I_x| = 2$ , and  $j_x = i_x$  if  $|I_x| = 1$ . Let  $\beta(x) = y_{i_x} - x$  and  $\gamma(x) = y_{j_x} - x$ . Since  $x \in V_{i_x}$  and  $x \in V_{j_x}$ , by (a) we have  $\frac{1}{2}\alpha(x) < \beta(x) < \alpha(x)$  and  $\frac{1}{2}\alpha(x) < \gamma(x) < \alpha(x)$ .

Let  $|I_x| = 1$ , i.e.  $I_x = \{i\}$ . Then  $i_x = i$  and  $g(x) = f(x, y_i) = f(x, x + \beta(x))$ . Now assume  $|I_x| = 2$ . Then  $I_x = \{i_x, j_x\}$  and

$$\begin{aligned} p(g(x) - f(x, x + \beta(x))) &= p\left(\sum_{i \in I_x} \varphi_i(x) f(x, y_i) - f(x, y_{i_x})\right) = \\ &= p(\varphi_{j_x}(x) f(x, y_{j_x}) - \varphi_{j_x}(x) f(x, y_{i_x})) \leq \\ &\leq p(f(x, y_{j_x}) - f(x, y_{i_x})) = p(f(x, x + \beta(x)) - f(x, x + \gamma(x))). \end{aligned}$$

□

**Theorem 3.2** Let  $X = \mathbb{R}$ , let  $Z$  be a metric linear space with the metric, generated by a pseudo-norm  $p$ , and let  $f : X^2 \rightarrow Z$  be a mapping which is continuous with respect to the first variable and has a finite variation and is continuous with respect to the second variable at every point of the diagonal  $\Delta = \{(x, x) : x \in X\}$ . Then  $g(x) = f(x, x)$  is an absolute Baire-one mapping.

**Proof.** For every  $n \in \mathbb{N}$  we apply Proposition 3.1 to  $f$  and to  $\alpha_n : X \rightarrow (0, +\infty)$ , where  $\alpha_n(x) = \frac{1}{2^n}$ . Then we obtain sequences  $(g_n)_{n=1}^\infty$ ,  $(\beta_n)_{n=1}^\infty$  and  $(\gamma_n)_{n=1}^\infty$  of continuous mappings  $g_n : X \rightarrow Z$  and functions  $\beta_n : X \rightarrow (\frac{1}{2^{n+1}}, \frac{1}{2^n})$  and  $\gamma_n : X \rightarrow (\frac{1}{2^{n+1}}, \frac{1}{2^n})$  such that

$$p(g_n(x) - f(x, x + \beta_n(x))) \leq p(f(x, x + \beta_n(x)) - f(x, x + \gamma_n(x)))$$

for any  $n \in \mathbb{N}$  and  $x \in X$ .

We show that the sequence  $(g_n)_{n=1}^\infty$  is as required. Fix  $x \in X$  and let  $z_n = g_n(x)$ ,  $u_n = f(x, x + \beta_n(x))$  and  $v_n = f(x, x + \gamma_n(x))$  for every  $n \in \mathbb{N}$ . Since  $\lim_{n \rightarrow \infty} \beta_n(x) = \lim_{n \rightarrow \infty} \gamma_n(x) = 0$  and  $f$  is continuous with respect to the second variable at  $(x, x)$ ,  $\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} v_n = f(x, x)$  and  $\lim_{n \rightarrow \infty} p(u_n - v_n) = 0$ . Now, taking into account that  $p(z_n - u_n) \leq p(u_n - v_n)$  for every  $n \in \mathbb{N}$ , we obtain that  $\lim_{n \rightarrow \infty} p(z_n - u_n) = 0$  and

$$\lim_{n \rightarrow \infty} g_n(x) = \lim_{n \rightarrow \infty} z_n = \lim_{n \rightarrow \infty} (z_n - u_n) + \lim_{n \rightarrow \infty} u_n = f(x, x) = g(x).$$

Remark that for every  $n \in \mathbb{N}$  the points  $x + \beta_n(x)$  and  $x + \gamma_n(x)$  belong to the interval  $I_n = (x + \frac{1}{2^{n+1}}, x + \frac{1}{2^n})$  and  $I_n \cap I_m = \emptyset$  for all distinct  $n$  and  $m$ . Moreover,  $\lim_{n \rightarrow \infty} (x + \beta_n(x)) = \lim_{n \rightarrow \infty} (x + \gamma_n(x)) = x$  and the mapping  $f^x : X \rightarrow Z$ ,  $f^x(t) = f(x, t)$ , has a finite variation at  $x$ . Hence,

$$\sum_{n=1}^{\infty} p(f^x(x + \beta_n(x)) - f^x(x + \gamma_n(x))) = \sum_{n=1}^{\infty} p(u_n - v_n) = C_1 < \infty$$

and

$$\sum_{n=1}^{\infty} p(f^x(x + \beta_{n+1}(x)) - f^x(x + \beta_n(x))) = \sum_{n=1}^{\infty} p(u_{n+1} - u_n) = C_2 < \infty.$$

Taking into account that  $p(z_n - u_n) \leq p(u_n - v_n)$  for all  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \sum_{n=1}^{\infty} p(g_{n+1}(x) - g_n(x)) &= \sum_{n=1}^{\infty} p(z_{n+1} - z_n) \\ &\leq \sum_{n=1}^{\infty} (p(z_{n+1} - u_{n+1}) + p(u_{n+1} - u_n) + p(z_n - u_n)) \leq \\ &\leq \sum_{n=2}^{\infty} p(u_n - v_n) + \sum_{n=1}^{\infty} p(u_{n+1} - u_n) + \sum_{n=1}^{\infty} p(u_n - v_n) \leq 2C_1 + C_2 < \infty. \end{aligned}$$

Thus,  $\sum_{n=1}^{\infty} |g_{n+1}(x) - g_n(x)|_Z < \infty$  and  $g$  is an absolute Baire-one mapping.  $\square$

**Corollary 3.3** *Let  $X \subseteq \mathbb{R}$  be an interval, let  $Z$  be a metric linear space with the metric, generated by a pseudo-norm, and let  $f : X^2 \rightarrow Z$  be a separately continuous mapping, which has finite variation with respect to the second variable at every point of the diagonal  $\Delta = \{(x, x) : x \in X\}$ . Then the mapping  $g : X \rightarrow Z$ ,  $g(x) = f(x, x)$ , is absolute Baire-one mapping.*

**Proof.** a). Let  $X = (a, b)$ , where  $a \in \mathbb{R} \cup \{-\infty\}$  and  $b \in \mathbb{R} \cup \{+\infty\}$ . Consider a homeomorphism  $\varphi : (a, b) \rightarrow \mathbb{R}$  such that  $\varphi$  and  $\varphi^{-1}$  are locally Lipschitz. Then the mapping  $\tilde{f} : \mathbb{R} \rightarrow Z$ ,  $\tilde{f}(t) = f(\varphi^{-1}(t))$ , satisfies the conditions of Theorem 3.2. Therefore, there exists a sequence  $(\tilde{g}_n)_{n=1}^\infty$  of continuous functions

$\tilde{g}_n : \mathbb{R} \rightarrow Z$  such that  $\lim_{n \rightarrow \infty} \tilde{g}_n(t) = \tilde{f}(t, t)$  and  $\sum_{n=1}^{\infty} |\tilde{g}_{n+1}(t) - \tilde{g}_n(t)|_Z < \infty$  for every  $t \in \mathbb{R}$ . Then the sequence  $(g_n)_{n=1}^{\infty}$  of continuous functions  $g_n : X \rightarrow Z$ ,  $g_n(x) = \tilde{g}_n(\varphi(x))$ , pointwise converges to  $g$  on  $X$  and  $\sum_{n=1}^{\infty} |g_{n+1}(x) - g_n(x)|_Z < \infty$  for every  $x \in X$ .

b). Now let  $X = [a, b]$ . According to a) there is a sequence  $(h_n)_{n=1}^{\infty}$  of continuous functions  $h_n : (a, b) \rightarrow Z$  such that  $\lim_{n \rightarrow \infty} h_n(x) = g(x)$  and  $\sum_{n=1}^{\infty} |h_{n+1}(x) - h_n(x)| < \infty$  for every  $x \in (a, b)$ . We choose a convergent to  $a$  monotone sequence  $(a_n)_{n=1}^{\infty}$  of reals  $a_n \in (a, b)$  and for every  $n \in \mathbb{N}$  let  $g_n : X \rightarrow Z$  be a continuous mapping such that  $g_n(a) = g(a)$  and  $g_n(x) = h_n(x)$  for every  $x \in [a_n, b]$ . Then  $(g_n)_{n=1}^{\infty}$  pointwise converges to  $g$  on  $X$  and  $\sum_{n=1}^{\infty} |g_{n+1}(x) - g_n(x)| < \infty$  for every  $x \in X$ .

c). In the case when  $X = (a, b]$  or  $X = [a, b]$  we argue analogously as we did to prove b).  $\square$

#### 4 The characterization of diagonals of separately absolutely continuous functions

**Proposition 4.1** *Let  $X$  be a metric space, let  $Z$  be a normed space, let  $f : X \rightarrow Z$  be a continuous mapping and  $\varepsilon > 0$ . Then there exists a locally Lipschitz mapping  $g : X \rightarrow Z$  such that  $\|f(x) - g(x)\| \leq \varepsilon$  for every  $x \in X$ .*

**Proof.** Since  $f$  is continuous and  $X$  is a paracompact by [4, Theorem 5.1.3], there exists a locally finite cover  $\mathcal{U}$  of  $X$  by open non-empty sets  $U$  with  $\text{diam } f(U) < \varepsilon$ . For every  $U \in \mathcal{U}$  and  $x \in X$  by  $\psi_U(x)$  we denote the distance from  $x$  to  $X \setminus U$ . Let  $\varphi_U = \frac{\psi_U}{\sum_{V \in \mathcal{U}} \psi_V}$  for every  $U \in \mathcal{U}$ . Note that  $(\varphi_U : U \in \mathcal{U})$  is a partition of unity on  $X$ , all the functions  $\varphi_U : X \rightarrow [0, 1]$  are locally Lipschitz and  $\text{supp } \varphi_U = U$ . Moreover, for every  $U \in \mathcal{U}$  we take  $z_U \in f(U)$ . Consider the function  $g : X \rightarrow Z$ ,  $g(x) = \sum_{U \in \mathcal{U}} \varphi_U(x) z_U$ . It is easy to see that  $\|f(x) - g(x)\| \leq \varepsilon$  for every  $x \in X$ . Since  $\mathcal{U}$  is locally finite and  $\varphi_U$  are locally Lipschitz,  $g$  is locally Lipschitz.  $\square$

The following example shows that the analogue of Proposition 4.1 for locally convex  $F$ -spaces  $Z$  is not valid.

**Example 4.2** *Let  $0 < p < 1$ , let  $X = \mathbb{R}$  be a space with Euclid metric, let  $Z = \mathbb{R}$  be a space with the metric  $|x - y|_Z = |x - y|^p$ ,  $f(x) = x$  for every  $x \in X$  and  $\varepsilon > 0$ . Since every locally Lipschitz mapping  $g : X \rightarrow Z$  is a constant,  $\sup\{|f(x) - g(x)|_p : x \in X\} > \varepsilon$  for any locally Lipschitz mapping  $g : X \rightarrow Z$ .*

**Proposition 4.3** Let  $g : [a, b] \rightarrow \mathbb{R}$  and  $h : [a, b] \rightarrow \mathbb{R}$  be Lipschitz functions with a constant  $K$ , let  $\varphi : [a, b] \rightarrow [0, 1]$  be a Lipschitz function with a constant  $L$  and  $|g(x) - h(x)| \leq M$  for every  $x \in [a, b]$ . Then the function

$$f(x) = \varphi(x)g(x) + (1 - \varphi(x))h(x)$$

is Lipschitz with the constant  $C = K + LM$  on  $[a, b]$ .

**Proof.** Let  $x_1, x_2 \in [a, b]$ . Then we have

$$\begin{aligned} |f(x_2) - f(x_1)| &= |(\varphi(x_2) - \varphi(x_1))(g(x_2) - h(x_2)) + \varphi(x_1)(g(x_2) - g(x_1)) + \\ &\quad + (1 - \varphi(x_1))(h(x_2) - h(x_1))| \leq LM|x_2 - x_1| + \varphi(x_1)K|x_2 - x_1| + \\ &\quad + (1 - \varphi(x_1))K|x_2 - x_1| = (K + LM)|x_2 - x_1|. \end{aligned}$$

□

**Proposition 4.4** Let  $Z$  be a metric space,  $f : [a, b] \rightarrow \mathbb{Z}$  a continuous mapping and let  $(a_n)_{n=1}^\infty$  be a strictly increasing sequence such that  $a_1 = a$ ,  $\lim_{n \rightarrow \infty} a_n = b$ , let for every  $n \in \mathbb{N}$  the mapping  $f$  is Lipschitz with a constant  $C_n$  on  $[a_n, a_{n+1}]$ , and moreover, the series  $\sum_{n=1}^\infty C_n(a_{n+1} - a_n)$  is convergent. Then  $f$  is absolutely continuous on  $[a, b]$ .

**Proof.** Fix  $\varepsilon > 0$  and choose  $m \in \mathbb{N}$  such that  $\sum_{n>m} C_n(a_{n+1} - a_n) < \frac{\varepsilon}{2}$ . Set  $\delta = \frac{\varepsilon}{2 \max\{C_n : 1 \leq n \leq m\}}$ . Let

$$\begin{aligned} a &\leq u_1 < v_1 \leq u_2 < v_2 \leq \cdots \leq u_i < v_i \leq a_m \leq \\ &\leq u_{i+1} < v_{i+1} \leq \cdots \leq u_{i+j} < v_{i+j} \leq b \end{aligned}$$

with  $\sum_{k=1}^{i+j} (v_k - u_k) < \delta$ . Then

$$\begin{aligned} \sum_{k=1}^{i+j} |f(v_k) - f(u_k)|_Z &= \sum_{k=1}^i |f(v_k) - f(u_k)|_Z + \sum_{k=i+1}^{i+j} |f(v_k) - f(u_k)|_Z \leq \\ &\leq \max\{C_n : 1 \leq n \leq m\} \cdot \sum_{k=1}^i (v_k - u_k) + \sum_{n>m} C_n(a_{n+1} - a_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Now let

$$a \leq u_1 < v_1 \leq \cdots \leq u_i < v_i \leq b$$

with  $\sum_{k=1}^i (v_k - u_k) < \delta$  and  $k \leq i$  such that  $a_m \in (u_k, v_k)$ . Using the above-obtained estimation we have

$$\sum_{j=1}^i |f(v_j) - f(u_j)|_Z = \sum_{j=1}^{k-1} |f(v_j) - f(u_j)|_Z + |f(a_m) - f(u_k)| +$$

$$|f(v_k) - f(a_m)| + \sum_{j=k+1}^i |f(v_j) - f(u_j)|_Z \leq \varepsilon.$$

□

**Theorem 4.5** *Let  $X \subseteq \mathbb{R}$  be an interval, let  $Z$  be a normed space and let  $g : X \rightarrow Z$  be an absolute Baire-one mapping. Then there exists a separately absolutely continuous mapping  $f : X^2 \rightarrow Z$  such that  $g(x) = f(x, x)$  for every  $x \in X$ .*

**Proof.** Let  $(I_n)_{n=1}^\infty$  be an increasing sequence of segments  $I_n = [a_n, b_n]$  such that  $X = \bigcup_{n=1}^\infty I_n$ . Let  $g_0(x) = 0$  for every  $x \in X$ . Using the definition of absolute Baire-one class and Proposition 4.1, we choose a sequence  $(\tilde{g}_n)_{n=1}^\infty$  of locally Lipschitz mappings  $\tilde{g}_n : X \rightarrow Z$  such that  $\lim_{n \rightarrow \infty} \tilde{g}_n(x) = g(x)$  and  $\sum_{n=1}^\infty \|\tilde{g}_{n+1}(x) - \tilde{g}_n(x)\| < \infty$  for every  $x \in X$ . Now for arbitrary  $n \in \mathbb{N}$  and  $x \in X$  let

$$g_n(x) = \begin{cases} \tilde{g}_n(x), & x \in I_n \\ \tilde{g}_n(a_n), & x \in X \cap (-\infty, a_n) \\ \tilde{g}_n(b_n), & x \in X \cap (b_n, +\infty). \end{cases}$$

Then we obtain the sequence  $(g_n)_{n=1}^\infty$  of Lipschitz mappings  $g_n : X \rightarrow Z$  such that  $\lim_{n \rightarrow \infty} g_n(x) = g(x)$  and  $\sum_{n=1}^\infty \|g_{n+1}(x) - g_n(x)\| < \infty$  for every  $x \in X$ .

Let  $(K_n)_{n=1}^\infty$  be a sequence of reals  $K_n > 0$  such that the functions  $g_{n-1}$  and  $g_n$  are Lipschitz with the constant  $K_n$ . We choose a strictly decreasing sequence  $(\delta_n)_{n=1}^\infty$  of reals  $\delta_n > 0$  such that  $\lim_{n \rightarrow \infty} \delta_n = 0$  and  $\sum_{n=1}^\infty K_n \delta_n < \infty$ .

For every  $n \in \mathbb{N}$  let  $F_n = \{(x, y) \in X^2 : |x - y| \leq \delta_n\}$ ,  $G_n = \{(x, y) \in X^2 : |x - y| < \delta_n\}$ . We consider the function  $\varphi_n : \mathbb{R} \rightarrow [0, 1]$  defined by

$$\varphi_n(t) = \begin{cases} 1, & |t| > \delta_n \\ \frac{t - \delta_{n+1}}{\delta_n - \delta_{n+1}}, & \delta_{n+1} \leq |t| \leq \delta_n \\ 0, & |t| < \delta_{n+1}. \end{cases}$$

Remark that every function  $\varphi_n$  is Lipschitz with the constant  $L_n = \frac{1}{\delta_n - \delta_{n+1}}$ .

We consider the following function  $f : X^2 \rightarrow Z$ :

$$f(x, y) = \begin{cases} g_0(x), & |x - y| > \delta_1 \\ \varphi_n(x - y)g_{n-1}(x) + (1 - \varphi_n(x - y))g_n(x), & (x, y) \in F_n \setminus F_{n+1} \\ g(x), & x = y \in X. \end{cases}$$



Let us show that  $f$  has the desired properties. Since for every  $n \in \mathbb{N}$  with  $|x - y| = \delta_{n+1}$  we have  $\varphi_n(x - y) = 0$  and  $\varphi_{n+1}(x - y) = 1$ ,

$$f(x, y) = g_n(x) = \varphi_n(x - y)g_{n-1}(x) + (1 - \varphi_n(x - y))g_n(x).$$

Therefore,

$$f(x, y) = \varphi_n(x - y)g_{n-1}(x) + (1 - \varphi_n(x - y))g_n(x)$$

for every  $(x, y) \in F_n \setminus G_{n+1}$ . The continuity of  $g_n$  and  $\varphi_n$  implies that  $f$  is jointly continuous on  $X^2 \setminus \Delta$ , where  $\Delta = \{(x, x) : x \in X\}$ . Moreover, the equality  $\lim_{n \rightarrow \infty} g_n(x) = g(x)$  for every  $x \in X$  implies that  $f$  is continuous with respect to the first and to the second variable at all points of  $\Delta$ . Thus,  $f$  is separately continuous. Moreover, since all of  $g_n$  and all of  $\varphi_n$  are Lipschitz,  $f$  is Lipschitz on  $X^2 \setminus G_n$  for every  $n \in \mathbb{N}$ .

We fix  $x_0 \in X$  and show that the mapping  $f_{x_0} : X \rightarrow Z$ ,  $f_{x_0}(x) = f(x, x_0)$ , is absolutely continuous. It is sufficient to prove that the mapping  $f_{x_0}$  is absolutely continuous on the intervals  $X_1 = X \cap (-\infty, x_0]$  and  $X_2 = X \cap [x_0, +\infty)$ . Let us consider the case when the interval  $X_1$  is non-degenerated, i.e.  $X_1 \cap (-\infty, x_0) \neq \emptyset$ . Since  $X = \bigcup_{n=1}^{\infty} I_n$  and  $(I_n)_{n=1}^{\infty}$  increases and  $\lim_{n \rightarrow \infty} \delta_n = 0$ , there exists a number  $m \in \mathbb{N}$  such that  $[x_0 - \delta_m, x_0] \subseteq I_m$ . Observe that  $f_{x_0}(x) = 0$  for every  $x \in X \cap (-\infty, x_0 - \delta_1]$  and  $f_{x_0}$  is Lipschitz on  $X_1 \cap [x_0 - \delta_1, x_0 - \delta_m]$ . Therefore, the absolute continuity of  $f_{x_0}$  of  $X_1$  is equivalent to the absolute continuity of  $f_{x_0}$  on  $[x_0 - \delta_m, x_0]$ .

Fix  $n \geq m$ . Notice that  $g_{n-1}$  and  $g_n$  are Lipschitz with the constant  $K_n$ , and  $\varphi_n$  is Lipschitz with the constant  $L_n$ . Moreover, for every  $x \in [x_0 - \delta_n, x_0 - \delta_{n+1}]$  we have

$$\begin{aligned} \|g_n(x) - g_{n-1}(x)\| &\leq \|g_n(x_0) - g_{n-1}(x_0)\| + \|g_n(x) - g_n(x_0)\| + \\ &+ \|g_{n-1}(x) - g_{n-1}(x_0)\| \leq \|g_n(x_0) - g_{n-1}(x_0)\| + 2K_n\delta_n = M_n. \end{aligned}$$

Hence,  $f_{x_0}$  is Lipschitz on  $[x_0 - \delta_n, x_0 - \delta_{n+1}]$  with the constant

$$C_n = K_n + L_n M_n = K_n + \frac{1}{\delta_n - \delta_{n+1}} (\|g_n(x_0) - g_{n-1}(x_0)\| + 2K_n\delta_n).$$

by Proposition 4.3. Now we have

$$\sum_{n \geq m} C_n(\delta_n - \delta_{n+1}) \leq 3 \sum_{n \geq m} K_n \delta_n + \sum_{n \geq m} \|g_n(x_0) - g_{n-1}(x_0)\| < \infty.$$

By Proposition 4.4,  $f_{x_0}$  is absolutely continuous in  $[x_0 - \delta_m, x_0]$ , consequently, on  $X_1$ .

The absolute continuity of  $f_{x_0}$  on  $X_2$  and the absolute continuity of  $f$  with respect to the second variable are proved similarly.  $\square$

Theorems 3.2 and 4.5 imply the following result.

**Theorem 4.6** *Let  $X \subseteq \mathbb{R}$  be an interval, let  $Z$  be a normed space and  $g : X \rightarrow Z$ . Then the following conditions are equivalent:*

- (i) *there exists a separately absolute continuous mapping  $f : X^2 \rightarrow Z$  with the diagonal  $g$ ;*
- (ii) *there exists a mapping  $f : X^2 \rightarrow Z$  with the diagonal  $g$  which is continuous with respect to the first variable and has a finite variation and is continuous with respect to the second variable at every point of the diagonal  $\Delta = \{(x, x) : x \in X\}$ ;*
- (iii)  *$g$  is an absolute Baire-one mapping.*

## 5 Open problems

**Question 5.1** *Let  $X = [0, 1]$ , let  $Z$  be a metric space and let  $f : X^2 \rightarrow Z$  be a separately absolute continuous mapping. Is  $g(x) = f(x, x)$  an absolute Baire-one mapping?*

**Question 5.2** *Let  $X = [0, 1]$ , let  $Z$  be a metric topological vector space and let  $g : X \rightarrow Z$  be an absolute Baire-one mapping. Does there exist a separately absolute continuous mapping  $f : X^2 \rightarrow Z$  such that  $g(x) = f(x, x)$  for every  $x \in X$ ?*

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